1.0

2.8　2.5

3.　2.2

1.1

2.0

1.8

1.25　1.4　1.6

AD A116214

MRC Technical Summary Report #2355

WATER WAVES IN A CHANNEL OF VARIABLE
CROSS SECTION

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March 1982

Received January 28, 1982

82 06 29 027

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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Xi-Chang Zhong[*] and M. C. Shen[†]

ABSTRACT

Based upon the ray theory, we develop a systematic method to obtain an equation of K-dV type with variable coefficients for the evolution of water waves in a channel of nonuniform cross section. Examples for channels with a nonuniform rectangular and triangular cross section are given. The fission of solitons in a triangular channel with a shoal is studied by the inverse scattering method and also numerically. A general Green's law for the decay of wave amplitude in a channel with arbitrary cross section is derived.

AMS(MOS) Subject Classification: 76B15, 76B25

Key Words: Ray theory, Water waves, channels, K-dV equation with
            variable coefficients, fission of solitons.

Work Unit No. 2 - Physical Mathematics

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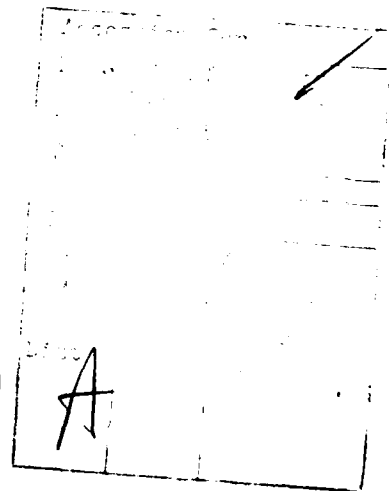53706 and the Computing Center, Chinese Academy of Sciences, Beijing, China.
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SIGNIFICANCE AND EXPLANATION

In this paper we propose a new equation for the study of nonlinear water waves in a channel of variable cross section. We also use both analytical and numerical methods to investigate the problem of a solitary wave climbing up a shoal in the triangular channel. A solitary wave is a wave with a single hump moving with a constant velocity. It is shown that under certain conditions the solitary wave may split into several similar waves. Furthermore, a general law to describe the attenuation of a wave in a channel is also derived.

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1. Introduction.

The K-dV equation originally derived for water waves over a uniform
bottom has been studied extensively since the inverse scattering method was
discovered by Gardner, Greene, Kruskal and Miura (1974) for the solution of
the equation. Recently there has been growing interest in K-dV equations with
variable coefficients, which appear in the study of water waves over a bottom
of nonuniform depth. The K-dV equation for a two-dimensional variable bottom
was derived by Kakutani (1971) and Johnson (1973), and that for a rectangular
channel with variable width and depth, by Shuto (1974). A review of the
recent developments regarding K-dV equations with variable coefficients may be
found in an article by Miles (1980). So far all the work on K-dV equations
for water waves over a variable bottom, one way or another, is related to the
equations for a rectangular channel, and the method of derivation cannot be
extended to channels with arbitrary nonuniform cross section. On the other
hand, the equations derived do not depend upon time explicitly, and conse-
quently an initial-value problem cannot be posed. Nevertheless, some inter-
esting phenomena have been observed from a study of these equations concerning
the effect of a perturbation on a solitary wave propagating in a channel of
variable depth. A single soliton may split into a finite number of solitons

if the depth of the channel decreases from one constant depth to another (Tappert and Zabusky, 1971; Johnson, 1973). Furthermore, the generation of a shelf is observed behind a solitary wave as it propagates in a rectangular channel with a slowly varying bottom as a perturbation about a constant depth (Ko and Kuell, 1978; Miles, 1979; Knickkerbocker and Newell, 1980).

The purpose of this paper is to propose a general K-dV equation, which applies to any channel with variable cross section. The method of derivation used here is essentially a specialization of the procedure developed by Shen and Keller (1973). Their work was motivated by the results due to Choquet-Bruhat (1969) on ray method for nonlinear partial differential equations and by Keller's linear ray expansion (1958) for water waves over a variable bottom. In terms of the time variable along a ray, the K-dV equation is now time-dependent and can be used to study initial value problems. Furthermore, there is great flexibility at our disposal to choose the appropriate independent variables in the K-dV equation, and a general Green's law for the decay of wave amplitude in a channel of variable cross section is also derived. Based upon the K-dV equation, we shall use inverse scattering method to study the fission of solitons as a solitary wave climbs up a shoal in a triangular channel. The results are then confirmed by numerical methods.

In Section 2 we formulate the problem and derive the general K-dV equation by means of the ray method. In Section 3 we specialize our result to the equations for rectangular and triangular channels. The analytical and numerical results for the fission of solitons in a triangular channel are given in Section 4, where the change of the channel width or depth is considered. In Section 5, a discussion of the results is given and the general Green's law is derived.

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1. Introduction.

The K-dV equation originally derived for water waves over a uniform bottom has been studied extensively since the inverse scattering method was discovered by Gardner, Greene, Kruskal and Miura (1974) for the solution of the equation. Recently there has been growing interest in K-dV equations with variable coefficients, which appear in the study of water waves over a bottom of nonuniform depth. The K-dV equation for a two-dimensional variable bottom was derived by Kakutani (1971) and Johnson (1973), and that for a rectangular channel with variable width and depth, by Shuto (1974). A review of the recent developments regarding K-dV equations with variable coefficients may be found in an article by Miles (1980). So far all the work on K-dV equations for water waves over a variable bottom, one way or another, is related to the equations for a rectangular channel, and the method of derivation cannot be extended to channels with arbitrary nonuniform cross section. On the other hand, the equations derived do not depend upon time explicitly, and consequently an initial-value problem cannot be posed. Nevertheless, some interesting phenomena have been observed from a study of these equations concerning the effect of a perturbation on a solitary wave propagating in a channel of variable depth. A single soliton may split into a finite number of solitons

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2. Derivation of the K-dV equation.

We consider the motion of an inviscid, incompressible fluid of constant density under gravity in a channel with a boundary defined by $h^*(x^*, y^*, z^*) = 0$, where z^* is positive upward (Figure 1). The governing equations are

$$u^*_{x^*} + v^*_{y^*} + w^*_{z^*} = 0 , \tag{1}$$

$$u^*_{t^*} + u^* u^*_{x^*} + v^* u^*_{y^*} + w^* u^*_{y^*} = -p^*_{x^*}/\rho , \tag{2}$$

$$v^*_{t^*} + u^* v^*_{x^*} + v^* v^*_{y^*} + w^* v^*_{z^*} = -p^*_{y^*}/\rho , \tag{3}$$

$$w^*_{t^*} + u^* w^*_{x^*} + v^* w^*_{y^*} + w^* w^*_{z^*} = -p^*_{z^*}/\rho - g , \tag{4}$$

subject to the boundary conditions

$$\left. \begin{array}{r} \eta^*_{t^*} + u^* \eta^*_{x^*} + v^* \eta^*_{y^*} - w^* = 0 \\[2mm] p^* = 0 \end{array} \right\} \quad \text{at} \quad z^* = \eta^*(x^*, y^*, t^*), \tag{5}\tag{6}$$

$$u^* h^*_{x^*} + v^* h^*_{y^*} + w^* h^*_{z^*} = 0 \quad \text{at} \quad h^* = 0. \tag{7}$$

Here (u^*, v^*, w^*) is the velocity, t^* is the time, g is the constant gravitational acceleration, ρ is the constant density, p^* is the pressure, and $z^* = \eta^*$ is the equation for the free surface. To derive the K-dV equation for the wave amplitude, we make the following assumptions. The channel bottom varies slowly in the longitudinal direction and the magnitude of the transverse velocities is much smaller than that of the longitudinal velocity. Needless to say, within the framework of long wave approximation different scalings used may give rise to different equations. However, in the following we shall be only concerned with the derivation of the K-dV equation. Based upon the assumptions of the slow variation of the channel bottom and different orders of magnitude of the velocities in different directions, we introduce the nondimensional variables:
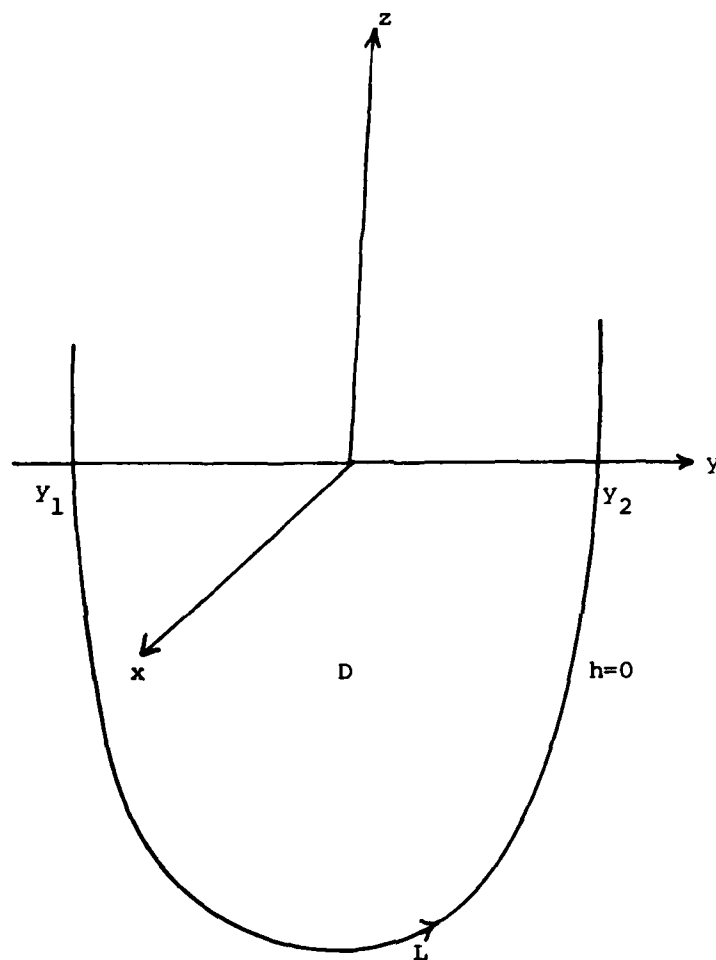
Figure 1

A cross section of the channel

$$t = \beta^{-3/2} t^* / (H/g)^{1/2}, \qquad (x,y,z) = (\beta^{-3/2} x^*/H, \ y^*/H, \ z^*/H),$$

$$\eta = \eta^*/H, \qquad h = h^*/H, \qquad p = p^*/(\rho g H).$$

$$(u,v,w) = (u^*/(gH)^{1/2}, \ \beta^{1/2} v^*/(gH)^{1/2}, \ \beta^{1/2} w^*/(gH)^{1/2}),$$

$$\beta^{3/2} = L/H \gg 1,$$

where L, H are respectively the longitudinal and transverse length scales. In terms of them, (1) to (7) become

$$u_x + \beta(v_y + w_z) = 0, \tag{8}$$

$$u_t + u u_x + p_x + \beta(v u_y + w u_z) = 0, \tag{9}$$

$$v_t + u v_x + \beta(v v_y + w v_z) + \beta^2 p_y = 0, \tag{10}$$

$$w_t + u w_x + \beta(v w_y + w w_z) + \beta^2(p_z + 1) = 0, \tag{11}$$

subject to the boundary conditions

$$\left. \begin{array}{l} \eta_t + u \eta_x + \beta(v \eta_y - w) = 0 \qquad \qquad (12) \\[2mm] p = 0 \qquad \qquad \qquad \qquad \qquad \qquad (13) \end{array} \right\} \quad \text{at} \quad z = \eta,$$

$$u h_x + \beta(v h_y + w h_z) = 0 \qquad \text{at} \quad h = 0. \tag{14}$$

We assume that u, v, w, p and η, as functions of t, x, y and z, also depends explicitly upon a new variable

$$\xi = \beta \ S(t,x)$$

where S will be called a phase function, and that they possess an asymptotic expansion of the form

$$\phi(\xi, t, x, y, z, \beta) \sim \phi_0 + \beta^{-1} \phi_1 + \beta^{-2} \phi_2 + \cdots . \tag{15}$$

Substitution of (15) in (8) to (14) will yield a sequence of equations and boundary conditions for the successive approximations by equating to zero the coefficients of the like powers of β. The solution for the zeroth approximation is assumed to be given, and for simplicity we assume the following one:

$$(u_0, v_0, w_0) = 0, \qquad p_0 = -z, \quad \eta_0 = 0. \tag{16}$$

The equations for the first approximation are

$$ku_{1\xi} + v_{1y} + w_{1z} = 0, \tag{17}$$

$$-\omega u_{1\xi} + kp_{1\xi} = 0, \tag{18}$$

$$p_{1y} = p_{1z} = 0, \tag{19}$$

subject to the boundary conditions

$$\left. \begin{array}{l} \omega\eta_{1\xi} + w_1 = 0 \\[1em] p_1 = \eta_1 \end{array} \right\} \quad \text{at} \quad z = 0 \qquad \begin{array}{l} (20) \\[1em] (21) \end{array}$$

$$v_1 h_y + w_1 h_z = 0 \qquad \text{at} \quad h = 0, \tag{22}$$

where $k = S_x$, $\omega = -S_t$. It is obtained from (19) that p_1 is a function of ξ, t and x only. We express

$$p_1 = A(\xi, t, x), \tag{23}$$

and from (17) and (18), it follows that

$$v_{1y} + w_{1z} = -A_\xi k^2/\omega. \tag{24}$$

Upon integrating both sides of (24) over a cross section D of the fluid domain for constant x, making use of divergence theorem and (20) to (22), we obtain

$$A_\xi [k^2 a(x) - \omega^2 b(x)] = 0, \tag{25}$$

where $a(x)$ is the area of D and $b(x)$ is the width of D in the plane $z = 0$. Assume that A_ξ is not identically zero, then (25) implies

$$\omega = kG(x), \quad G(x) = \pm[a(x)/b(x)]^{1/2}, \tag{26}$$

where $a(x)/b(x)$ is the mean depth of the cross section D. Equation (26) may be solved by means of the method of characteristics (Courant and Hilbert, 1962) and the corresponding characteristic equations are

$$\left. \begin{array}{l} dt/d\sigma = \mu, \quad dx/d\sigma = \mu G(x), \quad dk/d\sigma = -k\mu G'(x) \\[1.5em] d\omega/d\sigma = dS/d\sigma = 0, \end{array} \right\} \tag{27}$$

where μ is a proportionality factor. The solutions of (27) determine a one-

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parameter family of bicharacteristics, called rays,

$$t = t(\sigma, \sigma_1), \qquad x = x(\sigma, \sigma_1),$$

where σ_1 is constant along each bicharacteristic. We may choose $\mu = 1$ so that $\sigma = t$ and $x = x(t, \sigma_1)$. We also note that as seen from (27) both ω and S are also constant along a bicharacteristic.

The equations for the second approximation are

$$k\, u_{2\xi} + v_{2y} + w_{2z} + u_{1x} = 0, \tag{28}$$

$$-\omega\, u_{2\xi} + k u_1 u_{1\xi} + v_1 u_{1y} + w_1 u_{1z} + k p_{2\xi} + u_{1t} + p_{1x} = 0, \tag{29}$$

$$p_{2y} = \omega\, v_{1\xi}, \tag{30}$$

$$p_{2z} = \omega\, w_{1\xi}, \tag{31}$$

subject to the boundary conditions

$$\left. \begin{aligned} -\omega \eta_{2\xi} + k\, u_1 \eta_{1\xi} - w_2 - w_{1z}\, \eta_1 + \eta_{1t} = 0 \\ p_2 = \eta_2 \end{aligned} \right\} \quad \text{at} \quad z = 0, \tag{32}$$
$$\tag{33}$$

$$v_2 h_y + w_2 h_z = -u_1 h_x \qquad \text{at} \quad h = 0. \tag{34}$$

Differentiating (30) and (31) in turn with respect to y and z, adding and making use of (17) and (18), we obtain

$$\nabla^2 p_2 = -k^2 p_{1\xi\xi}. \tag{35}$$

It also follows from (20) to (22), (30) and (31) that

$$p_{2z} = -\omega^2 p_{1\xi\xi} \qquad \text{at} \quad z = 0 \tag{36}$$

$$p_{2y} h_y + p_{2z} h_z = 0 \qquad \text{at} \quad h = 0. \tag{37}$$

We may define

$$p_2 = -\phi\, p_{1\xi\xi} + A_2, \tag{38}$$

where

$$\phi = \phi(t, x, y, z), \qquad A_2 = A_2(\xi, t, x).$$

By (35) to (38), ϕ satisfies

$$\nabla^2 \phi = k^2, \tag{39}$$

$$\phi_z = \omega^2 \qquad \qquad \text{at} \quad z = 0, \tag{40}$$

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$$\phi_y h_y + \phi_z h_z = 0 \qquad \text{at} \quad h = 0. \tag{41}$$

Equations (39) to (41) pose a Neumann problem for ϕ, and the condition of solvability is seen to be satisfied as a consequence of (26). In terms of ϕ and A, we may determine (u_1, v_1, w_1) by integrating (18), (30) and (31), and assuming $(u_1, v_1, w_1) \to 0$, $p_1, p_{1\xi} \to 0$ as $\xi \to \infty$. This implies

$$(u_1, v_1, w_1) = \omega^{-1}(kp_1, -\phi_y p_{1\xi}, -\phi_z p_{1\xi}). \tag{42}$$

We remark here that other boundary conditions could be prescribed and the final equation for A would be inhomogeneous.

Now we are in a position to derive the K-dV equation for A. From (28) and (29),

$$V_{2y} + W_{2z} = -(k/w)(kp_{2\xi} + ku_1 u_{1\xi} + v_1 u_{1y} + w_1 u_{1z} + u_{1t} + p_{1x}) - u_{1x}. \tag{43}$$

Upon integrating both sides of (43) over a cross section D, making use of (26), (32) to (34) and (38) to (42), we finally obtain

$$m_0 A_t + m_1 A_x + m_2 A + m_3 A A_\xi + m_4 A_{\xi\xi\xi} = 0, \tag{44}$$

where

$$\left.\begin{aligned}
&m_0 = 2b(x), \qquad m_1 = 2\,a(x)/G(x), \\[2mm]
&m_2 = -(G^{-1}(x)\int_L h_x(h_y^2 + h_z^2)^{-\frac{1}{2}}\,ds - G^{-2}(x)G'(x)a(x), \\[2mm]
&m_3 = -\omega^{-1}[\phi_y(t,x,y_2,0) - \phi_y(t,x,y,0)] + 3k\,G^{-1}(x)b(x) \\[2mm]
&m_4 = \omega^{-1}\iint_D (\nabla\phi)^2\,dy\,dz,
\end{aligned}\right\} \tag{45}$$

$\int_L h_x(h_y^2 + h_z^2)^{-\frac{1}{2}}\,ds$ is the line integral along the boundary $h = 0$ in the cross section D (Figure 1), (h_x, h_y, h_z) is in the outward normal direction to the boundary $h = 0$ and $y_2 - y_1 = b(x)$. The detailed derivation of (44) is deferred to the Appendix. As seen from (26) and (27), along a ray

$$d/d\sigma = \partial/\partial t + (dx/dt \; \partial/\partial x = \partial/\partial t + G(x) \; \partial/\partial x \;,$$

we express (44) in terms of σ and ξ as

$$2b(x)A_\sigma + m_2 A + m_3 AA_\xi + m_4 A_{\xi\xi} = 0 \;, \tag{46}$$

which may be used to pose an initial value problem by prescribing initial data

at $\sigma = 0$.

3. Special cases.

In this section we shall explicitly derive the K-dV equations for rectangular and triangular channels with variable cross section, and show that all previous results are special cases of (44).

Case 1. Rectangular channel with variable cross section.

Assume that the two vertical walls of the rectangular channel are given by $y = -b_1(x)$, $y = b_2(x)$ and the depth by $z = -d(x)$, (Figure 2). We also let $b(x) = [b_2(x) + b_1(x)]$. The area for each cross section is $a(x) = b(x)d(x)$ and $G(x) = \pm d^{1/2}(x)$. To be definite, hereafter we shall only consider the plus sign for $G(x)$. It is easy to obtain from (45) that

$$m_0 = 2b(x), \qquad m_1 = 2b(x)d^{1/2}(x), \tag{47}$$

$$m_2 = b'(x)d^{1/2}(x) + d^{1/2}(x)d'(x)b(x)/2.$$

To determine m_3 and m_4, we need to find ϕ satisfying

$$\nabla^2 \phi = k^2 \, ,$$

$$\phi_z = \omega^2 \quad \text{at} \quad z = 0, \qquad \phi_z = 0 \quad \text{at} \quad z = -d(x),$$

$$\phi_y = 0 \quad \text{at} \quad y = -b_1(x), \quad y = b_2(x).$$

It is easily seen that $\phi_z = k^2(z + d)$, $\phi_y = 0$ where $k^2 d = \omega^2$. Hence, by (45) again we have

$$m_3 = 3k \, d^{-1/2}(x)b(x), \qquad m_4 = b(x)d^3(x)k^4/(3\omega). \tag{48}$$

Now we go back to the characteristic equations (27). By integrating $dx/dt = G(x) = d^{1/2}(x)$, the equations for rays are

$$\int_{x_0}^{x} G^{-1}(x)d\zeta = t - t_0 \, , \tag{49}$$

where (t_0, x_0) is the initial position of a ray. We choose $x_0 = 0$ and prescribe $S = -t$ on $x = 0$, then since $S = $ constant along each ray, we have

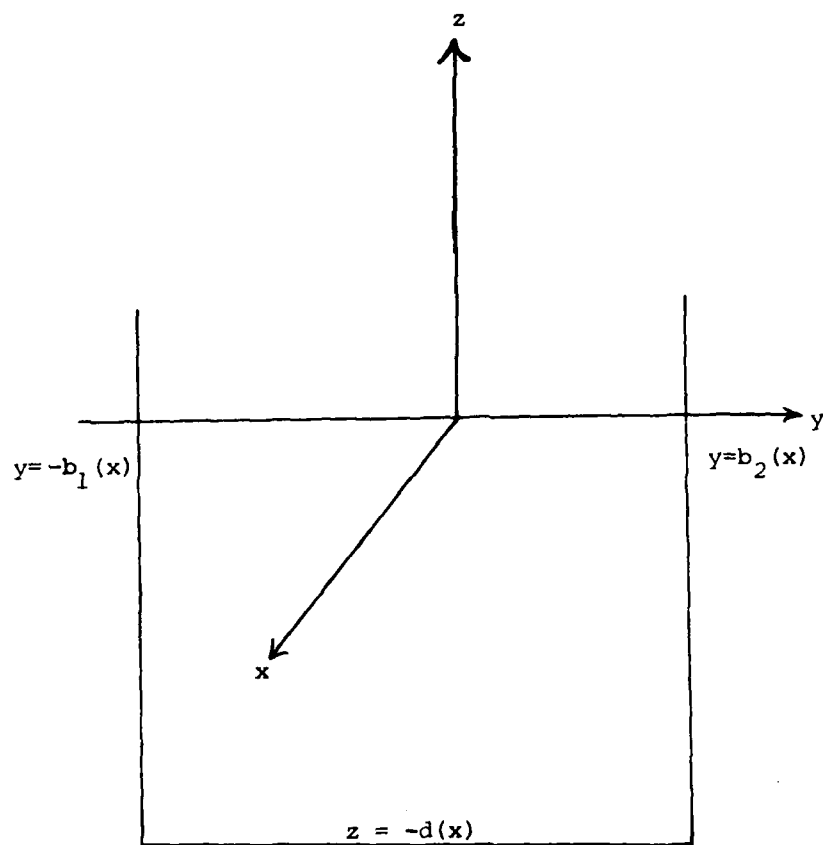$$S = -t_0 = -t + \int_0^x G^{-1}(\eta)d\eta. \tag{50}$$

Figure 2

A cross section of the rectangular channel

For this choice of S, S is time-like, and we obtain from (50) that

$$\omega = 1, \quad k = G^{-1}(x) = d^{-1/2}(x). \tag{51}$$

It follows from (44), (47), (48) and (51) that

$$2A_t + 2d^{1/2}(x)A_x + d^{-1/2}(x)[b'(x)b^{-1}(x)d(x) + d'(x)/2]A \tag{52}$$

$$+ 3^{d-1}(x)AA_\xi + d(x) A_{\xi\xi\xi}/3 = 0.$$

If we set $A_t = 0$ and assume that the channel is symmetric with respect to the plane $y = 0$, then apart from some scaling factor (52) reduces to the equation by Shuto (1974), and to that by Kakutani (1971) and Johnson (1973) when $b'(x) = 0$. In terms of σ as the time along the ray, (52) becomes

$$2A_\sigma + d^{-1/2}(x)[b'(x) b^{-1}(x)d(x) + d'(x)/2]A$$

$$+ 3 d^{-1}(x)AA_\xi + d(x)A_{\xi\xi\xi}/3 = 0 \tag{53}$$

where x is related to σ and ξ by (50)

$$\beta^{-1}\xi = -\sigma + \int_0^x d^{-1/2}(\eta)d\eta .$$

Case 2. Triangular channel with variable cross section.

The two sides of a cross section D of the triangular channel are given by $z = -\mu_1(x)y - d(x)$, $z = \mu_2(x)y - d(x)$ where $\mu_i(x) = d(x)/b_i(x)$, $i = 1,2$ (Figure 3). The area of D is $b(x)d(x)/2$ and $G(x) = d^{1/2}(x) \sqrt{2}$. We find from (45) that

$$\left. \begin{array}{l} m_0 = 2b(x) = 2[b_1(x) + b_2(x)], \quad m_1 = \sqrt{2}\, d^{1/2}(x)b(x), \\[2em] m_2 = \sqrt{2}\, d^{-1/2}(x)[b'(x)d(x) + d'(x)b(x)/2]/2. \end{array} \right\} \tag{54}$$

It is also easy to verify that

$$\phi = (k^2/4)[y^2 + (z + d(x))^2], \tag{55}$$
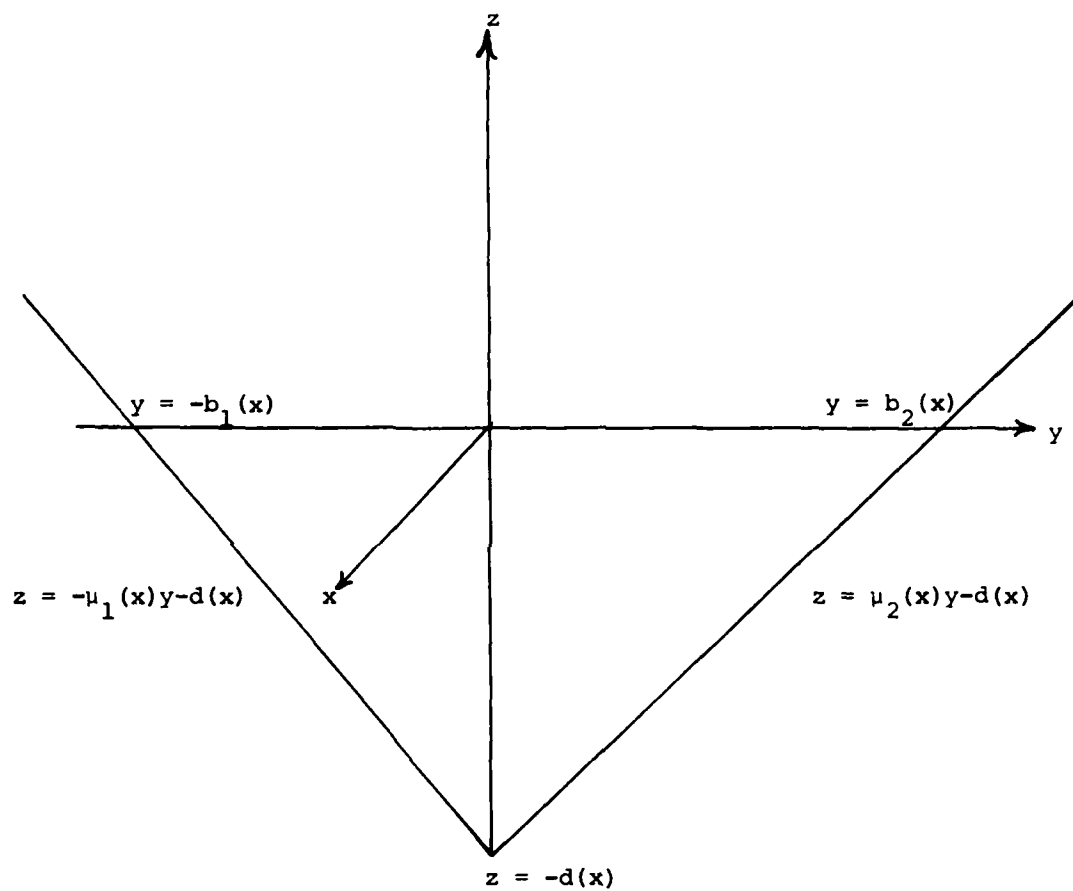
satisfies (39) to (41). Hence, by (45) and (55),

Figure 3

A cross section of the triangular channel

$$m_3 = (5\sqrt{2}/2)d^{-1/2}kb,$$

$$m_4 = (\sqrt{2}\ d^{-1/2}(x)k^3/16)[bd^3 + d(b_1^3 + b_2^3)/3].$$ (56)

If S is given by (50), then (46), (51), (54) and (56) imply

$$2A_\sigma + \sqrt{2}\ d^{-1/2}(x)[b'(x)b^{-1}(x)d(x) + d'(x)/2]A/2 + 5d^{-1}(x)AA_\xi$$

(57)

$$+ [d(x)/4 + (d^{-1}(x)/12)(b_1^2(x) - b_1(x)b_2(x) + b_2^2(x))]A_{\xi\xi\xi} = 0,$$

where $\omega = 1$, $k = \sqrt{2}\ d^{-1/2}$ and x is related to $\xi = \beta S$ and σ by (50).

4. Fission of solitons.

We consider a symmetric triangular channel with a cross section defined by

$$b(x) = b_0, \qquad d(x) = d_0, \qquad x \leqslant x_1$$

$$= b_1 \qquad\qquad = d_1, \qquad x > x_2,$$

where b_0, b_1, d_0, d_1 are constants and there is a transition zone in $x_1 \leqslant x \leqslant x_2$. For this problem, we may use x to replace σ as a variable along a ray. By (27),

$$dx/d\sigma = \beta(x) = d^{1/2}(x)/\sqrt{2}$$

and the K-dV equation for a triangular channel (57) may be expressed as

$$A_x + (\tfrac{1}{2})[b'(x)b^{-1}(x) + d'(x)(2d(x))^{-1}]A + (5\sqrt{2}/2)d^{-3/2}(x)AA_\xi \quad (58)$$

$$+ [\sqrt{2}/96)d^{-1/2}(x)[b^2(x)d^{-1}(x) + 12d(x)]A_{\xi\xi\xi} = 0,$$

where $b_1(x) = b_2(x)$ for a symmetric channel. We set

$$A = A^* b^{-1/2}(x) \ d^{-1/4}(x) \qquad\qquad (59)$$

and it follows from (58) that

$$A_x^* + (5\sqrt{2}/2)d^{-7/4}(x)b^{-1/2}(x)A^*A_\xi^* + (\sqrt{2}/96)d^{-1/2}(x)[b^2(x)d^{-1}(x) \quad (60)$$

$$+ 12d(x)]A_{\xi\xi\xi}^* = 0.$$

Let us now consider a solitary wave moving from $x = -\infty$ toward $x = +\infty$ as a progressive solution of (60) for $b(x) = b_0$, $d(x) = d_0$. We would like to find the conditions under which fission of solitons may take place after the solitary wave moves into the section where $b(x) = b_1$, $d(x) = d_1$. The basic ideas used in our approach are essentially due to Tappert and Zabusky (1971), Johnson (1973) and Miura et. al. (1974). Assume that both $d(x)$ and $b(x)$ are constant, and we transform (60) to the standard form of the K-dV equation

$$U_\zeta + 6UU_\xi + U_{\xi\xi\xi} = 0 \tag{61}$$

by introducing

$$U = p\,A^*/q, \qquad \zeta = qx/6 , \tag{62}$$

where

$$p = (5\sqrt{2}/2)d^{-7/4}(x)b^{-\frac{1}{2}}(x), \quad q/6 = (\sqrt{2}/96)d^{-\frac{1}{2}}(x)[b^2(x)d^{-1}(x) + 12d(x)]. \tag{63}$$

A progressive wave solution for U is given by

$$U = (c/2)\text{sech}^2[(c)^{\frac{1}{2}}(\xi - c\zeta)/2] . \tag{64}$$

In other words, if we prescribe U at $\zeta = 0$ in the form of (64), then U will be a solution for all ζ. Suppose that the transition of the channel takes place at $x = x_1 = 0$, and we prescribe

$$A^* = A_0^*\,\text{sech}^2\alpha\xi \qquad \text{at} \quad x = 0, \tag{65}$$

and from (59),

$$A = A_0^* b_0^{-\frac{1}{2}} d_0^{-\frac{1}{2}} \text{sech}^2 \alpha\xi \quad \text{at} \quad x = 0 .$$

Then

$$U = (p_0 A^*/q_0)\text{sech}^2 \alpha\xi \qquad \text{at} \quad \zeta = 0, \tag{66}$$

where

$$p_0 = (5\sqrt{2}/2)d_0^{-7/4}b_0^{-\frac{1}{2}}, \quad q_0/6 = (\sqrt{2}/96)d_0^{-\frac{1}{2}}(b_0^2 d_0^{-1} + 12d_0). \tag{67}$$

In comparing (65) with (64), A^* and α must satisfy the following conditions

$$p_0 A_0^*/q_0 = c/2, \quad \sqrt{c}/2 = \alpha ,$$

so that (65) is the initial condition for a progressive wave solution of (61) with p_0, q_0 given by (67). It follows that

$$p_0 A_0^*/(2q_0\,\alpha^2) = 1. \tag{68}$$

After the solitary wave moves into the section of the channel with $b(x) = b_1$, $d(x) = d_1$, we assume that the K-dV equation (61) with (63) given by

$$p_1 = (5\sqrt{2}/\alpha)d_1^{-7/4}b_1^{-\frac{1}{2}}, \quad q_1/6 = (\sqrt{2}/96)d_1^{-\frac{1}{2}}[b_1^2 d_1^{-1} + 12d_1], \tag{69}$$

could be solved by the inverse scattering method (Muira et. al., 1974) and consider the eigenvalue problem

$$d^2\psi/d\xi^2 + (U(0,\xi) + \lambda)\psi = 0,$$ (70)

where, by (62) and (65),

$$U(0,\xi) = (p_1 A_0^*/q_1)\text{sech}^2\alpha\xi.$$

It follows from the known results (Landau and Lifschitz, 1958; Johnson, 1973; Muira et. al., 1974) that U will consist of n solitons for large x if

$$\left.\begin{array}{ll}
p_1 A_0^*/(q_1 \alpha^2) = n(n+1), & n = 1,2,\cdots, \\
\lambda_m/\alpha^2 = -m^2, & m = 1,2,\cdots,n ,
\end{array}\right\}$$ (71)

and the amplitude for the mth soliton is given by

$$U_m = -2\lambda_m \quad m = 1,2,\cdots,n$$

and by (62) and (71),

$$A_m^* = q_1 U_n/p_1 = 2A_0^* m^2/n(n+1), \quad m = 1,2,\cdots,n.$$ (72)

By eliminating A_0^*/α^2 from (68) and (71), we obtain

$$P_1 q_0/q_1 P_0 = n(n+1)/2, \quad n = 1,2,\cdots$$ (73)

which yields a relationship among b_0, d_0, b_1 and d_1 for the fission of n solitons to take place Making use of (67) and (69) and letting $\epsilon_i = b_i/d_i$, i = 0,1, we may express (73) as

$$[(\epsilon_0^2 + 12)/(\epsilon_1^2 + 12)](\epsilon_0/\epsilon_1)^{-\frac{1}{2}}(d_1/d_0)^{-11/4} = n(n+1)/2, \quad n = 1,2,\cdots.$$ (74)

In the following we consider three special cases upon which our numerical results will be based.

(1) $\qquad \epsilon_0 = \epsilon_1$.

From (74), we obtain

$$d_1/d_0 = [n(n+1)/2]^{-4/11} \quad n = 1,2,\cdots .$$ (75)

It is clear from (75) that d_1/d_0 must be less than one for the fission of n > 1 solitons to take place.

(2) $d_0 = d_1$.

(74) now yields

$$[(b_0/d_0)^2 + 12][(b_1/d_0)^2 + 12]^{-1}(b_1/b_0)^{-\frac{1}{2}} = n(n+1)/2 \ , \quad n = 1,2,3,\cdots . \quad (76)$$

For simplicity, assume $b_0 = d_0 = 1$. Then from (76), we obtain

$$13b_1^{-\frac{1}{2}}(12 + b_1^2)^{-1} = n(n+1)/2, \quad n = 1,2,3,\cdots . \quad (77)$$

Since the left hand side of (77) is a monotonically decreasing function of

b_1 and equal to one as $b_1 = 1$, all the roots of (77) must be less than one

as $n > 1$. Hence, the fission of solitons occurs when $b_1 < 1$.

(3) $b_0 = b_1$.

We have from (74) again that

$$[(d_0/b_0)^{-2} + 12][(d_1/b_0)^{-2} + 12]^{-1}(d_1/d_0)^{-9/4} = n(n+1)/2.$$

Assume $b_0 = d_0 = 1$. Then it follows from the above equation that

$$13d_1^{-\frac{1}{4}}(12 + d_1^2)^{-1} = n(n+1)/2, \quad n = 1,2,3,\cdots . \quad (78)$$

Equation (78) also shows that $d_1 < 1$ is a necessary condition for fission to

take place.

There are now several numerical methods available for the solution of the

K-dV equation. Since the physical models described by the K-dV equation

represent situations requiring computations for large time, any numerical

method proposed must meet at least two requirements. First, the method must

yield sufficiently accurate wave amplitudes for many time steps in the process

of computation. Secondly, since the position of a wave front is just as

important as the wave amplitude, the proposed method must be capable of

predicting the position of a wave front with minimal error. The first

requirement is easy to meet, if the method is conservative (Richtmeyer and

Morton, 1967). However, it is much more difficult to meet the second

requirement. In order to compare results, we shall use the partially

corrected second-order Adams-Bashforth scheme as well as the Hopscotch scheme
to solve (60) numerically for the three special cases considered before, and
we use $n = 3$ for all cases.

(1) $\qquad \varepsilon_0 = \varepsilon_1$.

The depth of the triangular channel is defined by

$$d(x) = \begin{cases} d_0, & x \leqslant x_1 = 0, \\ (2/3)(d_1-d_0)(x_2)^{-3}(x)^3 + (1/3)(d_1-d_0)(x_2)^{-2}(x)^2 + d_0, & x \leqslant x \leqslant x_2, \\ d_1, & x > x_2, \end{cases}$$

where we choose $b_0 = d_0 = 1$, $x_2 = 0.01$. From (75), we have $d_1 = 0.5212$.

For $x \leqslant x_1$, (60) becomes

$$A_x^* + (5\sqrt{2}/2)A^* A_\xi^* + (13\sqrt{2}/96)A_{\xi\xi\xi} = 0. \qquad (79)$$

We prescribe, at $x = 0$,

$$A^* = A_0^* \operatorname{sech}^2\alpha\xi. \qquad (81)$$

Then by (67) and (68), we have

$$A_0^* = 2q_0\alpha^2/p_0 = 13\ \alpha^2/20. \qquad (82)$$

It follows from (72) that the amplitudes of the three solitons are given by

$$A_m^* = 13m^2\alpha^2/120, \qquad m = 1,2,3.$$

If we choose $A_0^* = 1$, then $\alpha^2 = 20/13$, and

$$A_1^* = 1/6, \quad A_2^* = 2/3, \quad A_3^* = 3/2.$$

The numerical results are shown in Figure 4.

(2) $\qquad d_0 = d_1$.

We define the width of the triangular channel by

$$b(x) = \begin{cases} b_0, & x \leqslant x_1 = 0, \\ (2/3)(b_1-b_0)(x_2)^{-3}(x)^3 + (1/3)(b_1-b_0)(x_2)^{-2}(x)^2 + b_0, & x_1 \leqslant x \leqslant x_2, \\ b_1, & x > x_2, \end{cases}$$
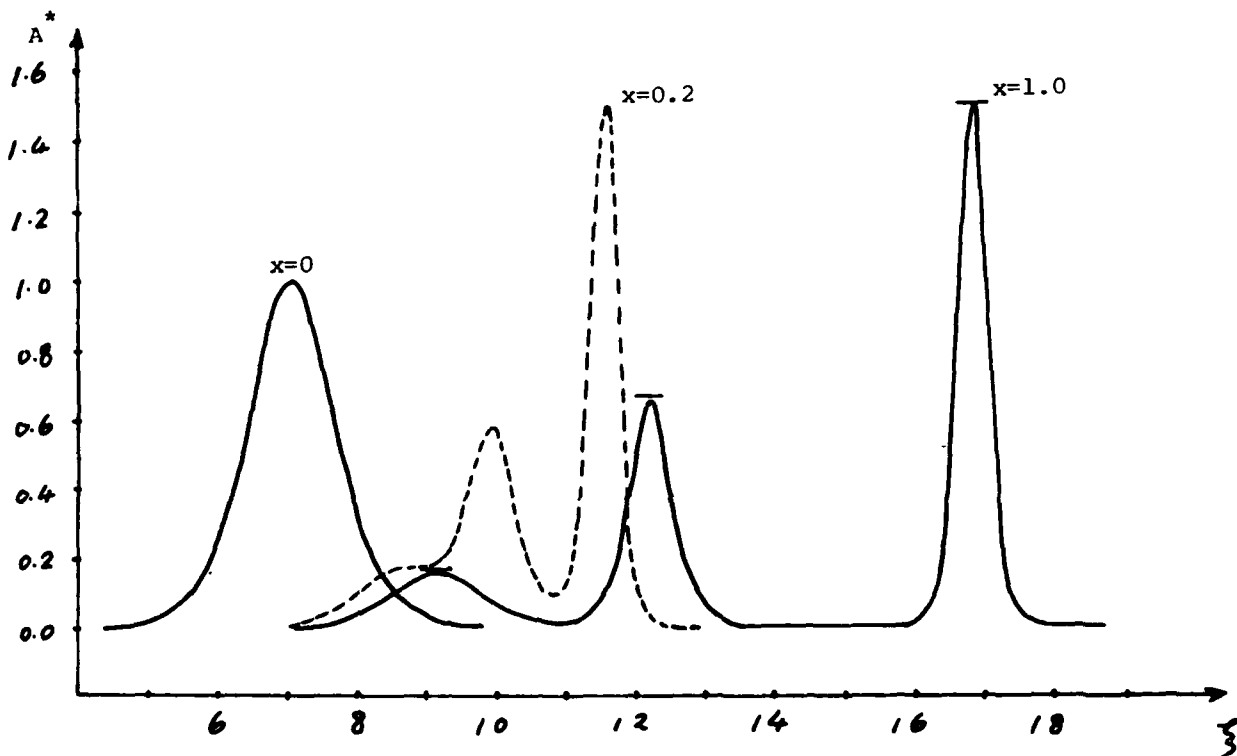
Figure 4.

Fission of solitons, $d_0 = b_0 = 1$, $d_1 = 0.5212$ (n=3) $c_0 = c_1$

where we also choose $b_0 = d_0 = 1$, and $x_2 = 0.01$. We obtain from (77) that $b_1 = 0.0326$. The K-dV equation for $x \leqslant 0$ is still (79). If we prescribe (81) with $A_0^* = 1$, $\alpha^2 = 20/13$ again, the amplitudes of the solitons are still given by (83) as shown in Figure 5.

(3) $b_0 = b_1$.

The depth function $d(x)$ is the same as in (1) with $b_0 = d_0 = 1$. From (78) $d_1 = 0.3824$. The amplitudes of the solitons remain the same, and the results are shown in Figure 6 if we choose $A^* = 1$, $\alpha^2 = 20/13$.

We note that the approximate method used here although confirmed by the numerical results, is still formal and its accuracy may depend upon the size of the transition zone. In a subsequent study we shall justify this method and show that, if $|x_2 - x_1|$ is sufficiently small, the solution obtained by the inverse scattering method is indeed an asymptotic approximation to the exact solution.
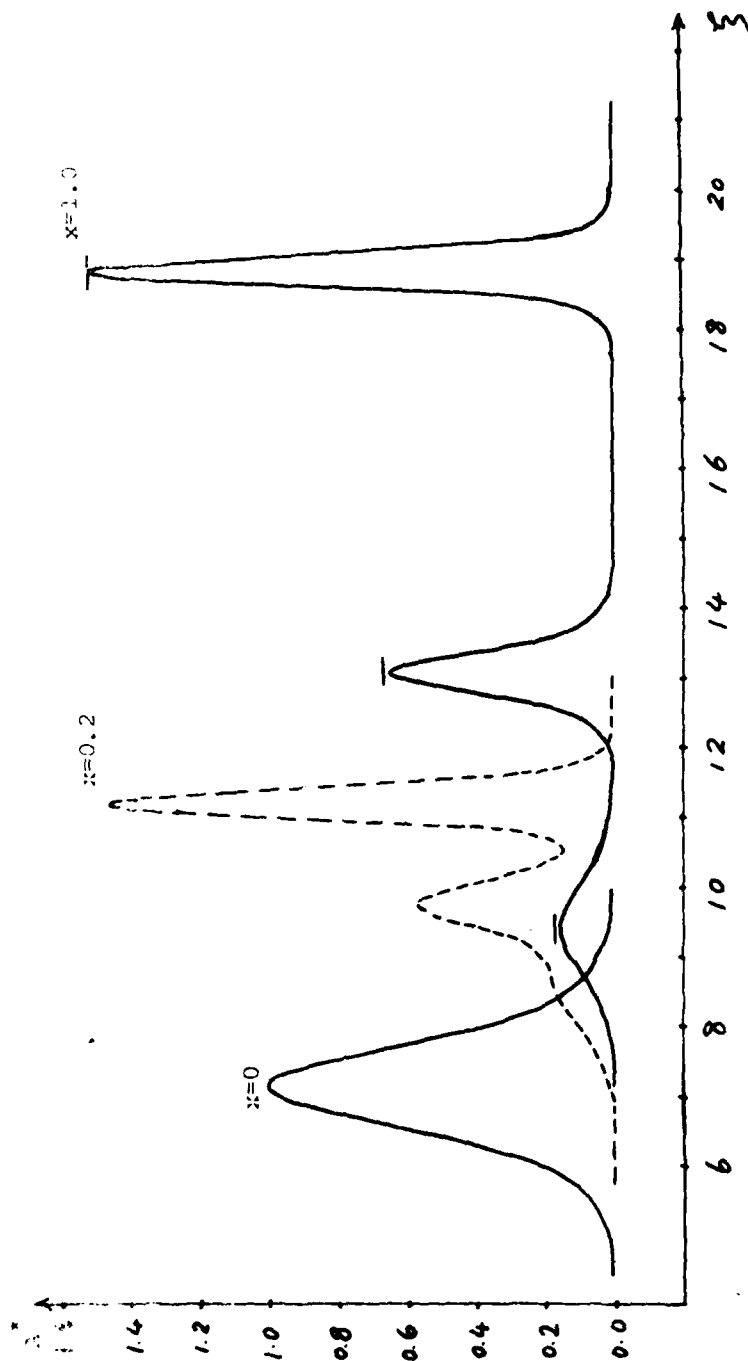
Figure 5

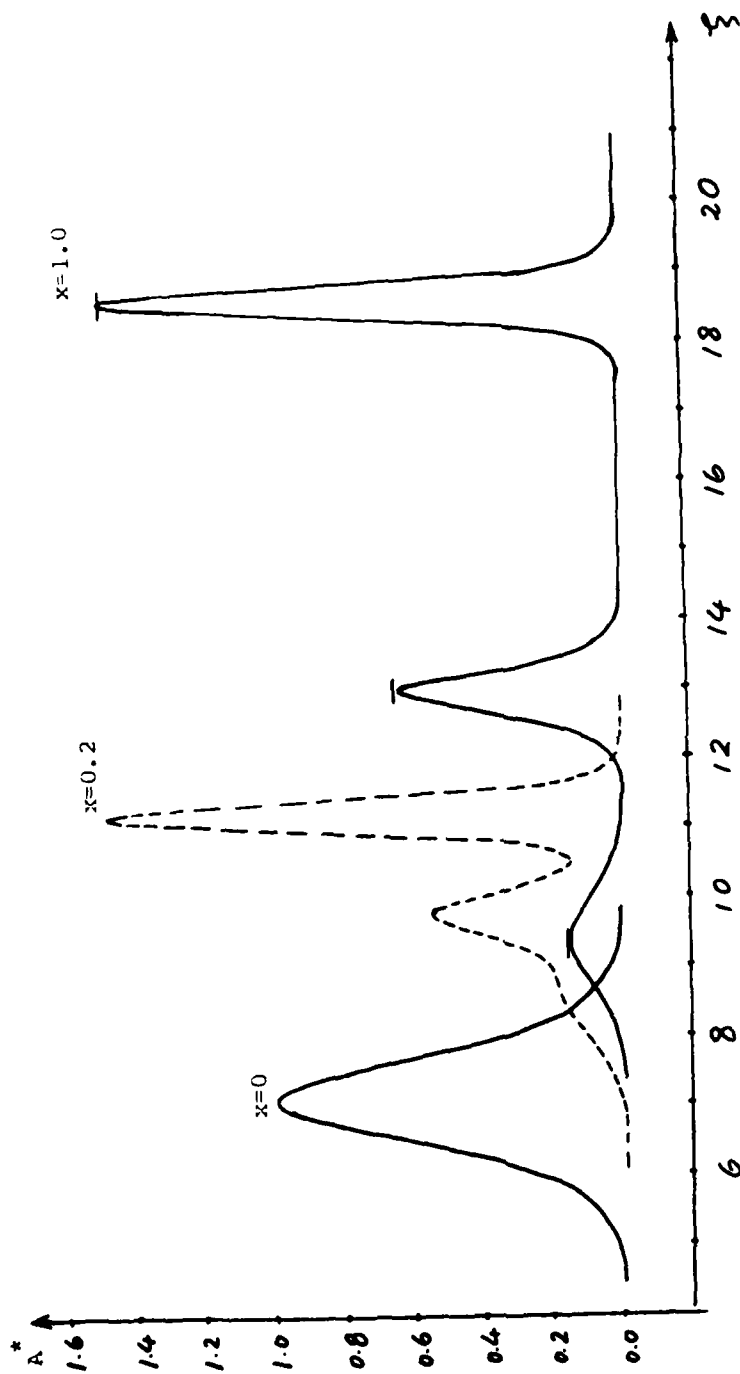Fission of Solitons, $d_0 = b_0 = d_1 = 1$, $b_1 = 0.0326$ (n=3)

Figure 6

Fission of Solitons. $d_0 = b_0 = b_1 = 1$, $d_1 = 0.3824$ *n = 3)

§5. Discussion.

In the derivation of the K-dV equation, we transformed the t, x-coordinates to the σ, σ_1-coordinates. We choose $\sigma = t$, $\sigma_1 = S$ since S is constant along a ray. Consider a simple smooth curve C: $t_0 = t_0(s)$, $x_0 = x_0(s)$, $-\infty < s < \infty$ in the t, x-plane, where we may identify S with s. Then by (49), S, as a function of t and x, is determined implictly by

$$\int_{x_0(S)}^{x} G^{-1}(\eta)d\eta = t - t_0(S). \tag{84}$$

The Jacobian of transformation from the t, S-coordinates to the t, x-coordinates is

$$J\left(\frac{t, x}{t, S}\right) = \partial x/\partial S = G(S)[G^{-1}(x_0)\, dx_0/dS - dt_0/dS].$$

Hence, $J \neq 0$ if $dx_0/dt_0 \neq G(x_0)$ where we assume $G(S) > 0$, and it follows that the rays do not intersect each other if C has no characteristic direction. Now we turn to (53) and (57). If we prescribe A as a function of x at $\sigma = 0$, we have to change x to ξ according to (50) where $S = \beta^{-1}\xi$. An alternative is to prescribe $S = x$ at $t = 0$. Then by (84), S is determined implicitly by

$$\int_{S}^{x} G^{-1}(\eta)d\eta = t.$$

In this case S is space-like and

$$\omega = G(S), \quad k = G(S)/G(x). \tag{85}$$

By (44), (48) and (85), we have

$$2A_\sigma + d^{-1/2}(x)[b'(x)b^{-1}(x)d(x) + d'(x)/2]A + 3d^{1/2}(S)d^{-1}(x)AA_\xi$$

$$+ d(x)d^{5/2}(S)A_{\xi\xi\xi}/3 = 0,$$

for a rectangular channel, and by (44), (54), (56) and (8), we obtain

$$2A_\sigma + \sqrt{2}d^{-1/2}(x)[b'(x)b^{-1}(x)d(x) + d'(x)/2]A/2$$

$$+ 5\, d^{-1}(x)d^{1/2}(S)\, AA_\xi + d^{3/2}(S)[d^{-1}(x)b^2(x) + 12d(x)]A_{\xi\xi\xi}/48 = 0,$$

for a triangular channel. In both cases, the initial data can be explicitly expressed in terms of $\beta^{-1}\xi$.

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In this work, we only consider the drivation of the K-dV equations. If we linearized the governing equations, and carried out the ray method, we would get the same Hamilton-Jacobian equation (26) and the following equation would hold:

$$m_0 A_t + m_1 A_x + m_2 A = 0, \qquad (86)$$

where m_0, m_1 and m_2 are given in (45). Along each ray, we may use x again as a variable and (8) assumes the form.

$$2b(x)G(x)A_x + m_2 A = 0 \qquad (87)$$

by (27) where $dx/dt = G(x)$. It follows from (8) that

$$\frac{d}{dx}\left\{A \exp \int^x m_2(\xi)(2b(\xi)G(\xi))^{-1} d\xi\right\} = 0,$$

along a ray, and

$$A = A_0 \exp[-\int_{x_0}^x m_2(\xi)(2b(\xi)G(\xi))^{-1} d\xi], \qquad (88)$$

where A_0 is the value of A at $x = x_0$, and

$$m_2(\xi) = -G^{-1}(\xi) \int_L h_\xi(\xi,y,z)(h_y^2(\xi,y,z) + h_z^2(\xi,y,z))^{-\frac{1}{2}} ds$$
$$- G^{-2}(\xi)G'(\xi)a(\xi).$$

Note that the attenuation factor in (88) only depends upon the geometry of the channel and may be considered as a generalization of the so-called "Green's law" for amplitude decay (Lamb 1932). For rectangular and triangular channels, we obtain from (47), (51), (54) and $G(x) = d^{\frac{1}{2}}(x)/\sqrt{2}$ for a triangular channel that the same Green's law holds:

$$A = A_0 b^{-\frac{1}{2}}(x)d^{-\frac{1}{4}}(x),$$

which is essentially (59).

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APPENDIX

We integrate both sides of (43) over a cross section D (Figure 1). By using divergence theorem, (32) and (34), we have

$$\iint_D (v_{2y} + w_{2z})dydz = -u_1 \int_\Gamma h_x(h_y^2+h_z^2)^{-\frac{1}{2}} ds + \int_{y_1}^{y_2} w_2|_{z=0}dy$$

$$= -u_1 \int_\Gamma h_x(h_y^2+h_z^2)^{-\frac{1}{2}} ds + \int_{y_1}^{y_2} [-\omega p_{2\xi} + ku_1\eta_{1\xi} - w_{1z}\eta_1 + \eta_{1t}]_{z=0}dy$$

$$= -\iint_D (k/w)(kp_{2\xi} + ku_1u_{1\xi} + u_{1t} + p_{1k}) + u_{1x}]dtdz$$

where $y_{1y} = u_{1z} = 0$. Now we make use of (21), (23), (26) and (38) to (42) to compute the coefficients m_0 to m_4 in (44).

$$m_0 = b(x) + k^2a(x)/\omega^2 = 2b(x),$$

$$m_1 = 2(k/\omega) a(x) = 2a(x)/G(x),$$

$$m_2 = -(k/\omega) \int_T h_x(h_y^2 + h_z^2)^{-\frac{1}{2}} ds + (k/\omega)(k/\omega)_t a(x) + (k/\omega)_x a(x)$$

$$= -G^{-1}(x) \int_\Gamma h_x(h_y^2 + h_z^2)^{-\frac{1}{2}} ds - G^{-2}(x)G'(x)a(x),$$

where $(k/\omega)_t = [G^{-1}(x)]_t = 0$,

$$m_3 = (k^2/\omega)b(x) + \omega^{-1} \int_{y_1}^{y_2} \phi_{zz}(t,x,y,0)dy + (k^4/\omega^3)a(x)$$

$$= 2kG^{-1}(x)b(x) + \omega^{-1} \int_{y_1}^{y_2} [k^2 - \phi_{yy}(t,x,y,0)] dy$$

$$= 3kG^{-1}(x)b(x) - \omega^{-1}[\phi_y(t,x,y_2,0) - \phi_y(t,x,y_1,0)].$$

$$m_4 = \omega \int_{y_1}^{y_2}\phi(t,x,y,0)dy - (k^2/\omega) \iint_D \phi dydz$$

$$= \omega \int_{y_1}^{y_2} \phi(t,x,y,0)dy - \omega^{-1} \iint_D \phi \nabla^2\phi dydz$$

$$= \omega \int_{y_1}^{y_2} \phi(t,x,y,0)dy - \omega^{-1}\int_{y_1}^{y_2}\omega^2 p(t,x,y,0)dy + \omega^{-1}\iint_D (\nabla\phi)^2 dydz$$

$$= \omega^{-1} \iint_D (\nabla\phi)^2 dydz .$$

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4. TITLE *(and Subtitle)* WATER WAVES IN A CHANNEL OF VARIABLE CROSS SECTION		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Xi-Chang Zhong and M. C. Shen		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS800-1960
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 2 - Physical Mathematics
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE March 1982
		13. NUMBER OF PAGES 28
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

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19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Water waves, channels, K-dV equation with variable coefficients, fission of solitons, Ray theory.

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*
    Based upon the ray theory, we develop a systematic method to obtain an equation of K-dV type with variable coefficients for the evolution of water waves in a channel of nonuniform cross section. Examples for channels with a nonuniform rectangular and triangular cross section are given. The fission of solitons in a triangular channel with a shoal is studied by the inverse scattering method and also numerically. A general Green's law for the decay of wave amplitude in a channel with arbitrary cross section is derived.

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